# A fresh look at the wave interference

immediate

June 4, 2025

## 1 The general analysis

Let us consider two coherent travelling periodic waves of the form:

$$E_1(r_1, t) = E_{m1}(r_1)\sin(\omega t - kr_1), \tag{1}$$

$$E_2(r_2, t) = E_{m2}(r_2)\sin(\omega t - kr_2), \tag{2}$$

where $r_1$ and $r_2$ are the distances from sources $s_1$ and $s_2$ to the observation point $M$; $\omega$ is the angular frequency; $k = 2\pi/\lambda$ is the wave number; $E_{m1,2}$ are the waves amplitudes at point $M$.

According to the superposition principle, the resulting wave is also travelling:

$$E = E_1 + E_2 = E_m(r_1, r_2)\sin(\omega t + \varphi_0(r_1, r_2)), \tag{3}$$

where

$$E_m = \sqrt{E_{m1}^2 + E_{m2}^2 + 2E_{m1}E_{m2}\cos\left(k|r_2 - r_1|\right)}, \tag{4}$$

$$\tan\varphi_0 = -\frac{E_{m1}\sin kr_1 + E_{m2}\sin kr_2}{E_{m1}\cos kr_1 + E_{m2}\cos kr_2}, \tag{5}$$

Here $E_m$ is the wave amplitude; $\varphi_0$ is the initial phase. Below we consider two particular cases.

## 2 The case of plane waves

In this case $E_{m1,2} = E_0 = \text{const}$. Then, equations (4) and (5) transforms to

$$E_m = 2E_0 \left| \cos\left(\frac{k|r_2 - r_1|}{2}\right) \right|, \tag{6}$$

$$\varphi_0 = \frac{k(r_1 + r_2)}{2}. \tag{7}$$

Considering equation (7), we conclude that condition of constant initial phase that defines the shape of the wave fronts is the following:

$$r_1 + r_2 = \text{const}. \tag{8}$$

Therefore, the wave fronts represent a set of confocal ellipsoids of revolution (figure 1). Wherein, point sources $s_1$ and $s_2$ are located at two common foci

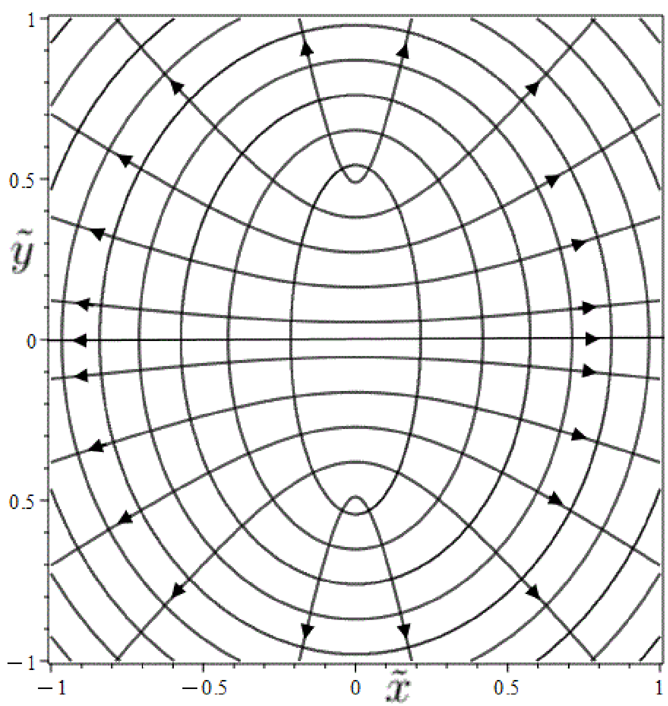


Figure 1: The wave fronts (confocal ellipses) and wave rays (confocal hyperbolas) of the resulting wave in the case of interference of two plane waves.

of these ellipsoids. It means that in the case of two plane wave interference the resulting travelling wave is an ellipsoidal wave.

An important property of the confocal ellipses and hyperbolas is that they form an orthogonal net of curves. Thus, the ellipsoidal waves propagate along confocal hyperbolas. According to equation (6), the intensity of the resulting wave remains constant precisely along these hyperbolas ($r_2 - r_1 = \text{const}$). The maximum irradiance (constructive interference) is achieved along those hyperbolas (bulge lines) that satisfy the condition:

$$|r_2 - r_1| = m\lambda, \tag{9}$$

where $m = 0, 1, 2, \ldots$ The minimum (zero) irradiance (destructive interference) corresponds to those hyperbolas (nodal lines) that satisfy the condition:

$$|r_2 - r_1| = (2m + 1)\,\frac{\lambda}{2}. \tag{10}$$

Since, for an arbitrary hyperbola $|r_2 - r_1| \leq d$, where $d$ is the distance between its foci, we have that the number of nodal lines $N_n$ is finite and equal to

$$N_n = 2\lceil d/\lambda - 1/2 \rceil, \tag{11}$$

where $\lceil z \rceil$ is the ceiling function. The number of bulge lines $N_b$ is found as

$$N_b = 2\lfloor d/\lambda \rfloor + 1, \tag{12}$$

where where $\lfloor z \rfloor$ is the floor function.

If $d \gg \lambda$, the number of the nodal and bulge lines can be very large. In the opposite case, when $d < \lambda/2$, there only one (bulge) line which is a straight line passing through the middle of the segment connecting the foci.

Finally, given the reflective property of an ellipse (the normal to any point of the ellipse forms equal acute angles with the focal radii of this point), we conclude that in the case of crossed plane waves, the energy transfer occurs along the bisector of the angle formed by them.

# 3 The case of spherical waves

In this case $E_{m1,2} = A/r_{1,2}$, where $A = \text{const}$. Using equations (4), (5), we find:

$$I = I_0 \left( \frac{1}{\tilde{r}_1^2} + \frac{1}{\tilde{r}_2^2} + \frac{2\cos\left(2\pi\alpha|\tilde{r}_2 - \tilde{r}_1|\right)}{\tilde{r}_1 \tilde{r}_2} \right), \tag{13}$$

$$\varphi_0 = -\arctan\left(\frac{\dfrac{\sin(2\pi\alpha\tilde{r}_1)}{\tilde{r}_1} + \dfrac{\sin(2\pi\alpha\tilde{r}_2)}{\tilde{r}_2}}{\dfrac{\cos(2\pi\alpha\tilde{r}_1)}{\tilde{r}_1} + \dfrac{\cos(2\pi\alpha\tilde{r}_2)}{\tilde{r}_2}}\right), \tag{14}$$

where $I = n\varepsilon_0 c E_m^2/2$ is the irradiance ($n$ is the refractive index of the medium of propagation; $\varepsilon_0$ is the vacuum permittivity; $c$ is is the speed of light in vacuum); $I_0 = n\varepsilon_0 c A^2/(2d^2)$ is the characteristic irradiance; $\alpha = d/\lambda$ is the characteristic parameter of the theory; $\tilde{r}_{1,2} = r_{1,2}/d$ are the relative (dimensionless) distances. If we introduce a two-dimensional Cartesian coordinate system $x - y$ such that the origin $O$ is located midway between the sources $s_1$ and $s_2$, and the $y$-axis passes through them, then

$$\tilde{r}_{1,2} = \sqrt{\tilde{x}^2 + \left(\tilde{y} \pm \frac{1}{2}\right)^2}, \tag{15}$$

where $\tilde{x} = x/d$, $\tilde{x} = x/d$ are the dimensionless coordinates.

Equations (13), (15) allows one to visualize the distribution of the irradiance in an arbitrary $x - y$ plane. In figure 2 we plot the set of contour lines of equal irradiance in the case of interference of two spherical waves. Close

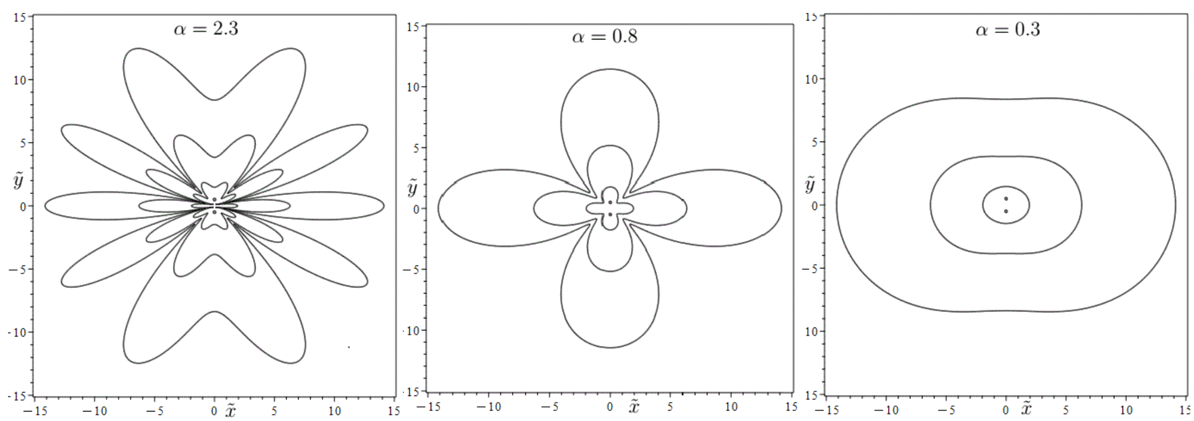


Figure 2: The set of four contour lines of equal irradiance $I$ in the case of interference of two spherical waves. Contours: $I = 100I_0$; $I = I_0$; $I = 0.1I_0$; $I = 0.02I_0$.

to the sources, these lines are nearly concentric, while at a sufficient distance from them, the lines form "rosettes with petals". As in the case of plane waves, the lines of local maxima and minima ($I = 0$) form hyperbolas, which are defined by equations (9) and (10). The only difference is that along the lines of local maxima, the irradiance does not remain constant but decreases proportionally to $(1/r_1 + 1/r_2)^2$. The lines of local maxima are the geometric locus of points corresponding to the peaks of the "petals". The lines of local minima are the geometric locus of points where the "petals" connect.

Figure 3 shows the set of contour lines of the same phase in the case of interference of two spherical waves. Close to the sources, these lines are

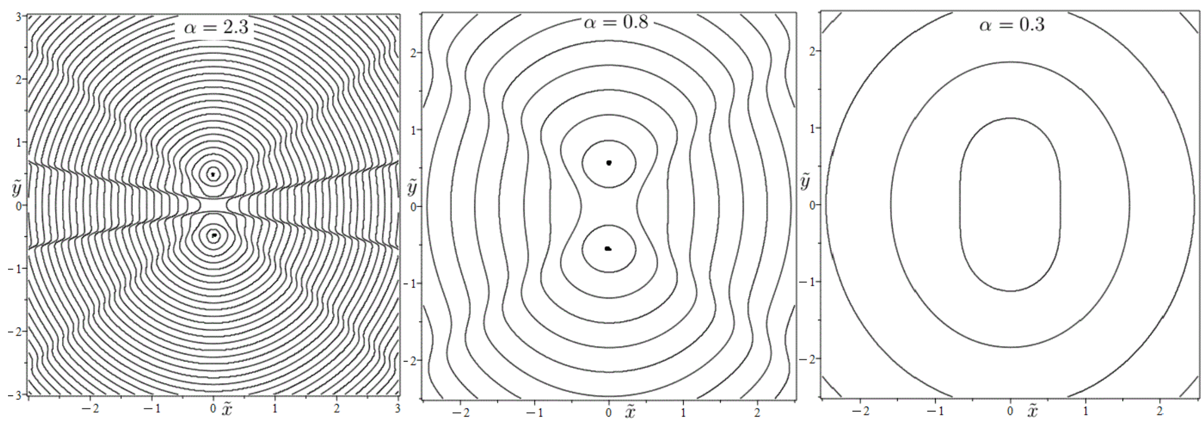


Figure 3: The set of contour lines of the equal initial phase $\varphi_0 = 0$ in the case of interference of two spherical waves.

nearly concentric, while at a sufficient distance from them, they are closed curves containing inflection points. The lines of the irradiance local minima are precisely the geometric locus of a set of inflection points. As $\alpha$ increases, the mesh of these contour lines becomes more "sparse".